SL Paper 1 Mock B 2021 – WORKED SOLUTIONS v1

Section A

1. Two points $P(-5,4)$ and $Q(11,-8)$ are on the line L, so L has gradient

$$m=\frac{4-(-8)}{-5-11}=\frac{12}{-16}=-\frac{3}{4}$$

The gradient of the line perpendicular to L is therefore $m_{\perp}=\frac{4}{3}$

The midpoint of $[PQ]$ is $\left(\frac{-5+11}{2},\frac{4+(-8)}{2}\right)\Rightarrow\left(\frac{6}{2},\frac{-4}{2}\right)\Rightarrow(3,-2)$

Solving for the equation of the line perpendicular to L:

$$y-(-2)=\frac{4}{3}(x-3)\Rightarrow y+2=\frac{4}{3}x-4\Rightarrow y=\frac{4}{3}x-6$$

Thus, the line perpendicular to L that passes through $(3,-2)$ has equation $y=\frac{4}{3}x-6$

2. (a) $h(x)=(f\circ g)(x)=f(g(x))$

$$f(g(x))=f(2x-3)$$

$$=\frac{1}{2(2x-3)+1}$$

$$=\frac{1}{4x-6+1}$$

$$f(g(x))=\frac{1}{4x-5}$$

thus, $h(x)=\frac{1}{4x-5}$

(b) $y=\frac{1}{4x-5}$; solve for x

$$\frac{1}{y}=4x-5\Rightarrow 4x=\frac{1}{y}+5\Rightarrow x=\frac{1}{4y}+\frac{5}{4}$$

thus, $h^{-1}(x)=\frac{1}{4x}+\frac{5}{4}$ or, equivalently, $h^{-1}(x)=\frac{5x+1}{4x}$

SL Paper 1 Mock B 2021 – WORKED SOLUTIONS v1

3. (a) $g'(x)=(x-4)^3$; find $g''(x)$ using the chain rule

let $u=x-4$ and let $y=u^3$

$$\frac{dy}{du}=3u^2,\quad \frac{du}{dx}=1$$

$$g''(x)=\frac{dy}{du}\cdot\frac{du}{dx}=3u^2\cdot 1$$

undo substitution:

$$g''(x)=3(x-4)^2$$

(b) $g''(4)=3(4-4)^2=0$

(c) $g''(x)\geq 0$ for all values of x, meaning that the graph of $g(x)$ is always concave up. An inflexion point requires a change in sign of the value of $g''(x)$, i.e. it requires the graph of $g(x)$ to go from concave up to concave down, or vice versa. As $g(x)$ is always concave up, neither point A nor any other point on $g(x)$ is an inflexion point.

4. (a) $\log_3 20=\log_3(4\cdot 5)=\log_3(2^2)+\log_3 5=2\log_3 2+\log_3 5=2x+y$

thus, $\log_3 20=2x+y$

(b) $\log_3\left(7\tfrac{13}{16}\right)=\log_3\left(\frac{7\cdot 16+13}{16}\right)=\log_3\left(\frac{112+13}{16}\right)=\log_3\left(\frac{125}{16}\right)$

$$\log_3\left(\frac{125}{16}\right)=\log_3 125-\log_3 16=\log_3(5^3)-\log_3(2^4)=3\log_3 5-4\log_3 2=3y-4x$$

thus, $\log_3\left(7\tfrac{13}{16}\right)=3y-4x$

(c) $\log_5 8=\dfrac{\log_3 8}{\log_3 5}=\dfrac{\log_3(2^3)}{\log_3 5}=\dfrac{3\log_3 2}{\log_3 5}=\dfrac{3x}{y}$

thus, $\log_5 8=\dfrac{3x}{y}$

SL Paper 1 Mock B 2021 – WORKED SOLUTIONS v1

5. (a) $1+\ln x+(\ln x)^2+\cdots$ is a geometric sequence with $u_1=1$ and $r=\ln x$

a geometric sequence converges when $|r|<1$

$|\ln x|<1 \Rightarrow -1<\ln x<1$

$\ln x<1 \Rightarrow x<\mathrm{e}$

$\ln x>-1 \Rightarrow x>\mathrm{e}^{-1}$

thus, the sequence converges when $\frac{1}{\mathrm{e}}<x<\mathrm{e}$

(b) the sum of an infinite geometric sequence is $S_\infty=\frac{u_1}{1-r}$

for this sequence, $u_1=1$ and $r=\ln x$

$S_\infty=\frac{1}{1-\ln x}$; we want to find the value of x such that $S_\infty=2$

$\frac{1}{1-\ln x}=2 \Rightarrow 1-\ln x=\frac{1}{2} \Rightarrow \ln x=\frac{1}{2} \Rightarrow x=\mathrm{e}^{\frac{1}{2}}=\sqrt{\mathrm{e}}$

thus, the sequence converges to 2 when $x=\sqrt{\mathrm{e}}$

6. $8\sin x\cos x=\sqrt{12}$; simplify LHS using trigonometric identities

$\text{LHS}=8\sin x\cos x=4(2\sin x\cos x)=4\sin 2x$

simplify RHS:

$\text{RHS}=\sqrt{12}=\sqrt{4\cdot 3}=2\sqrt{3}$

$4\sin 2x=2\sqrt{3} \Rightarrow \sin 2x=\frac{\sqrt{3}}{2} \Rightarrow 2x=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

we are told to solve the equation for $0\le x\le\frac{\pi}{2}$, i.e. $0\le 2x\le\pi$

therefore, $2x=\frac{\pi}{3}, \frac{2\pi}{3}$

hence, $x=\frac{\pi}{6}, \frac{\pi}{3}$

SL Paper 1 Mock B 2021 – WORKED SOLUTIONS v1

Section B

7. (a) $\frac{1}{8}+p=1 \Rightarrow p=\frac{7}{8}$

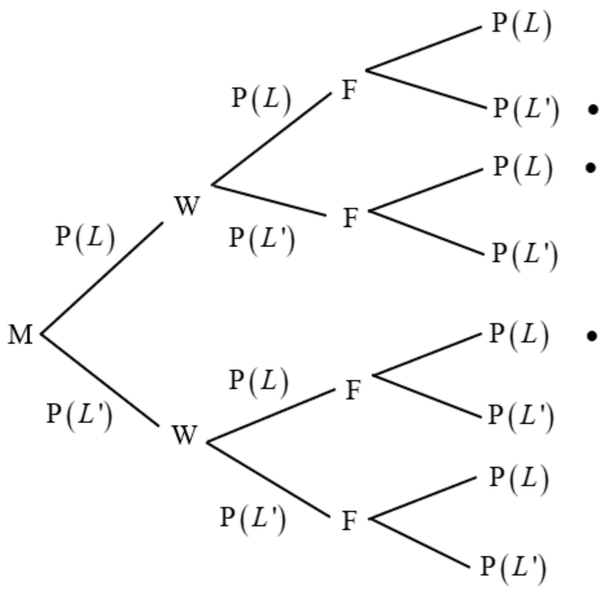
(b) $P(T \cap L)=\frac{2}{3}\cdot\frac{1}{8}=\frac{1}{12}$

(c) $P(L)=\frac{2}{3}\cdot\frac{1}{8}+\frac{1}{3}\cdot\frac{1}{4}=\frac{1}{12}+\frac{1}{12}=\frac{1}{6}$

(d) $P(T \mid L)=\frac{P(T \cap L)}{P(L)}=\frac{\frac{1}{12}}{\frac{1}{6}}=\frac{1}{2}$

(e) on any given school day, the probability that Sophie will be late is $P(L)=\frac{1}{6}$

construct a tree diagram of the three days that Sophie goes to school, with M for Monday, W for Wednesday and F for Friday. Each day, there is a probability of being late, $P(L)$, and a probability of not being late, $P(L')$.



Out of the 8 possible outcomes, 3 are such that Sophie is late for school exactly twice in the week. These are labelled in the tree diagram above by a black circle. Therefore, the probability that Sophie is late for school exactly twice next week is the probability of any of these 3 outcomes occurring.

Each of the 3 outcomes have an equal probability of occurring, which is $P(L)\cdot P(L)\cdot P(L')$

So, the probability of Sophie being late exactly twice next week is $3\left(P(L)\cdot P(L)\cdot P(L')\right)$

Substituting $P(L)=\frac{1}{6}$ and $P(L')=\frac{5}{6}$:

$$3\left(\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}\right)=\frac{1}{2}\cdot\frac{5}{36}=\frac{5}{72}$$

Thus, the probability that Sophie will be late exactly twice next week is $\frac{5}{72}$.

Alternatively, the answer can be found directly as a binomial distribution where $X \sim B\left(3,\frac{1}{6}\right)$.

Then, $P(X=2)=\binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)=\frac{5}{72}\approx 0.0694$

SL Paper 1 Mock B 2021 – WORKED SOLUTIONS v1

8. (a) $f(x) = 2x^2 + 4x - 5 = 2(x^2 + 2x) - 5$; completing the square

$f(x) = 2(x^2 + 2x + 1) - 5 - 2$

$f(x) = 2(x+1)^2 - 7$

(b) (i) $(-1, -7)$

(ii) $x = -1$

(iii) $(0, -5)$

(iv) x-intercepts exist where $f(x) = 0$

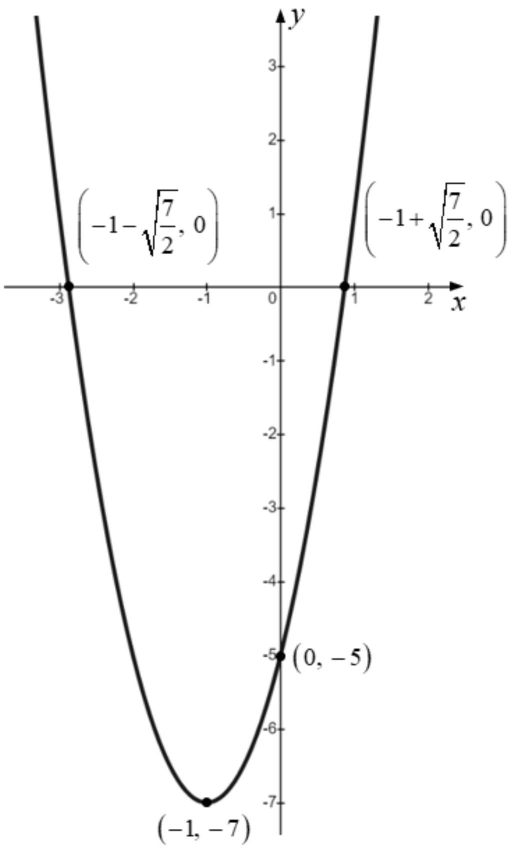
$2(x+1)^2 - 7 = 0 \;\Rightarrow\; 2(x+1)^2 = 7$

$(x+1)^2 = \frac{7}{2} \;\Rightarrow\; x + 1 = \pm\sqrt{\frac{7}{2}}$

$x = -1 \pm \sqrt{\frac{7}{2}}$

thus, x-intercepts are $x = -1 + \sqrt{\frac{7}{2}}$ and $x = -1 - \sqrt{\frac{7}{2}}$

(c)



$\sqrt{\frac{7}{2}} < \sqrt{\frac{8}{2}} = \sqrt{4} = 2$

thus, $\sqrt{\frac{7}{2}}$ is slightly less than 2

hence, it follows that $x = -1 + \sqrt{\frac{7}{2}}$ is slightly less than 1, and

$x = -1 - \sqrt{\frac{7}{2}}$ is slightly greater than -3

SL Paper 1 Mock B 2021 – WORKED SOLUTIONS v1

9. (a) $g(x)=\dfrac{3x}{x^2+7}$; find $g'(x)$ using quotient rule

let $u=3x \Rightarrow \dfrac{du}{dx}=3$

let $v=x^2+7 \Rightarrow \dfrac{dv}{dx}=2x$

$$g'(x)=\frac{3\cdot(x^2+7)-3x\cdot 2x}{(x^2+7)^2}$$

$$=\frac{3x^2+21-6x^2}{(x^2+7)^2}$$

$$g'(x)=\frac{21-3x^2}{(x^2+7)^2}$$ **Q.E.D.**

(b) $\displaystyle\int\frac{3x}{x^2+7}dx$; using integration by substitution

let $u=x^2+7 \Rightarrow \dfrac{du}{dx}=2x$

$$\int\frac{3x}{x^2+7}dx=\frac{3}{2}\int\frac{1}{u}du$$

$=\dfrac{3}{2}\ln u+C$; undo substitution

$$\int\frac{3x}{x^2+7}dx=\frac{3}{2}\ln(x^2+7)+C$$

(c) area $=\displaystyle\int_{\sqrt{7}}^{a}g(x)dx$

$$=\int_{\sqrt{7}}^{a}\frac{3x}{x^2+7}dx$$

from (b), $\displaystyle\int\frac{3x}{x^2+7}dx=\frac{3}{2}\ln(x^2+7)+C$

SL Paper 1 Mock B 2021 – WORKED SOLUTIONS v1

thus, area $=\left[\frac{3}{2}\ln\left(x^2+7\right)\right]_{\sqrt{7}}^{a}$

$=\frac{3}{2}\ln\left(a^2+7\right)-\frac{3}{2}\ln\left(\left(\sqrt{7}\right)^2+7\right)$

$=\frac{3}{2}\ln\left(a^2+7\right)-\frac{3}{2}\ln 14$

$=\frac{3}{2}\ln\left(\frac{a^2+7}{14}\right)$

we are given that the area of the region is $\ln 8$

hence, $\frac{3}{2}\ln\left(\frac{a^2+7}{14}\right)=\ln 8$

$\ln\left(\frac{a^2+7}{14}\right)=\frac{2}{3}\ln 8=\ln\left(8^{\frac{2}{3}}\right)==\ln\left(\left(2^3\right)^{\frac{2}{3}}\right)=\ln\left(2^2\right)$

$\ln\left(\frac{a^2+7}{14}\right)=\ln 4$

$\frac{a^2+7}{14}=4 \Rightarrow a^2+7=56$

$a^2=56-7=49 \Rightarrow a=\pm 7$

$a>\sqrt{7}$, therefore $a=7$